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LETTER TO THE EDITOR

Oscillator representations of the 2D conformal algebra and superalgebra

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Abstract. For the Virasoro algebra and for the Neveu-Schwarz-Ramond (NSR) superalgebra, oscillator representations are found by localising the Miura and the super Miura maps from the theory of the Korteweg-de Vries (KdV) and the super KdV equations, respectively.

The history of science is full of unexpected links between seemingly unconnected theories. One of the most illustrious examples is provided by the dual resonance theory and the theory of Kac-Moody Lie algebras. Both were formulated in 1968 (Veneziano 1968, Kac 1968, Moody 1968), both have independently undergone spectacular developments and they have cross-fertilised each other ever since 1980 when a deep similarity between their formal structures was found (Frenkel and Kac 1980).

A year later in 1981 (Drinfel'd and Sokolov 1981), Kac-Moody algebras were found to serve as one of the underlying structures of another important subject: the theory of infinite-dimensional integrable systems, proposed in 1967 (Gardner *et al* 1967), one year earlier than the above two theories. It would be, then, not unreasonable to suspect the existence of a hidden connection between integrable systems and dual models, and an indication that such a connection does indeed exist is the appearance of the Virasoro algebra both in dual models (Schwarz 1973) and in the description of the Korteweg-de Vries (KdV) equation (Kupershmidt 1985a).

In this letter I show how to use one of the central objects in the theory of integrable systems—the Miura map—to find new oscillator representations for the Virasoro algebra and for the Neveu-Schwarz-Ramond (NSR) superalgebra. The method used—localisation—can be applied to any Hamiltonian structure (e.g. that of classical or quantum fluids) and to any Hamiltonian map between a pair of Hamiltonian structures (e.g. for Clebsch maps associated with representations of functional Lie algebras (see Kupershmidt 1985b, ch VIII)).

Recall (Kupershmidt and Wilson 1981) that the second Hamiltonian structure B^2 of the KdV equation

$$\dot{u} = B^2(\delta H/\delta u) \qquad B^2 = u\partial + \partial u + c\partial^3 \qquad \partial = \partial/\partial x \qquad 0 \neq c \in \mathbb{C} \tag{1}$$

and the Hamiltonian structure B of the KdV equation,

$$\dot{v} = B\left(\frac{\delta H}{\delta v}\right) \qquad B = ca^{-1}\partial \qquad 0 \neq a \in \mathbb{C} \tag{2}$$

are related by the Miura map

$$u = av_x - a^2c^{-1}v^2/2 \tag{3}$$

which is Hamiltonian (= canonical). We now rewrite (1)-(3) in terms of the coefficients $\{u_n\}$ and $\{v_n\}$ of the Taylor series of u and v respectively. Denote by $C_u = \mathbb{C}[u^{(j)}]$ $j \in \mathbb{Z}_+$ and $C_v = \mathbb{C}[v^{(j)}]$ $j \in \mathbb{Z}_+$, the corresponding differential algebras, and let

$$T\colon C_u \to \hat{C}_u \qquad C_v \to \hat{C}_v \qquad T(P) = \sum_{n\in\mathbb{Z}} T(P)_n x^n \tag{4}$$

be the Taylor map, given on generators by the rule

$$T(u^{(j)}) = \sum n_n(\partial_x)^j(x^n) \qquad T(v^{(j)}) = \sum v_n(\partial_x)^j(x^n) \tag{5}$$

where

$$\hat{C}_u = \prod_{k=-\infty}^{\infty} \mathbb{C}[u_n]x^k, n\in\mathbb{Z} \qquad \text{and} \qquad \hat{C}_v = \prod_{k=-\infty}^{\infty} \mathbb{C}[v_n]x^k, n\in\mathbb{Z},$$

are the completions of the corresponding polynomial rings (see Manin 1979, § I.7.2.1). Denote

$$\tilde{P} = T(P)_{-1} \qquad P \in C_u \text{ or } C_v. \tag{6}$$

Using arguments from infinite-dimensional variational calculus, similar to those in Manin (1979 § I.7.22), one can show that

$$T(\delta P/\delta u)_n = \frac{\partial \tilde{P}}{\partial u_{-1-n}} \qquad P \in C_u, n \in \mathbb{Z} \tag{7}$$

and analogously for $P \in C_v$ (Manin works with the Fourier representation

$$F(u^{(j)}) = \sum_{n\in\mathbb{Z}} (2\pi i n)^j u_n \exp(2\pi i n x)$$

instead of our Taylor representation; the analogue of formula (7) in the Fourier case is his Lemma I.7.22: $F(\delta P/\delta u)_n = \partial\tilde{P}/\partial u_{-n}$, $\tilde{P} = F(P)_0$. Using (5)-(7), and denoting

$$L_n = u_{-2+n} \qquad a_n = v_{-1+n} \tag{8}$$

we obtain from (1)-(3)

$$\dot{L}_n = cn(n^2-1)\frac{\partial \tilde{H}}{\partial L_{-n}} + \sum (n-m)L_{(n+m)}\frac{\partial H}{\partial L_m} \tag{9}$$

$$\dot{a}_n = -ca^{-2}n\frac{\partial H}{\partial a_{-n}} \tag{10}$$

$$L_n = a(n-1)a_n - a^2c^{-1}\sum_{l+m=n} a_l a_m/2. \tag{11}$$

Thus, the Hamiltonian matrices of (9) and (10) are

$$B^2_{nm} = (n-m)L_{(n+m)} + cn(n^2-1)\delta_{(n,-m)} \tag{12}$$

$$B_{nm} = -ca^{-2}n\delta_{(n,-m)}. \tag{13}$$

In the RHS of (12) we recognise the commutator $[L_n, L_m]$ of the Virasoro algebra, on the dual space of which lives the matrix B^2 in (12). The map (11) is Hamiltonian between (12) and (13); this follows from the fact that (3) is Hamiltonian and the abstract Hamiltonian formalism; alternatively, it is easy to check this fact by a direct calculation. Rewriting (12) and (13) once again in the form of the basic Poisson brackets

$$\{L_n, L_m\} = (n-m)L_{(n+m)} + cn(n^2-1)\delta_{(n,-m)} \tag{14}$$

$$\{a_n, a_m\} = ca^{-2}m\delta\delta_{(n,-m)} \tag{15}$$

we can interpret (15) as giving us a set of free oscillators $\{a_n\}$ and (11) as providing us with an oscillator representation of the Virasoro algebra (14). In contrast to the known oscillator representations of the Virasoro algebra (Gervais and Neveu 1985), there are no ordered products involved in (11); in addition, the form of the expression (11) is uniform in n, i.e. it does not depend upon whether $n=0$.

We conclude by applying the same idea to the super KdV situation (Kupershmidt 1984). Here

$$\begin{pmatrix}\dot{u}\\ \dot{\varphi}\end{pmatrix} = B^2\begin{pmatrix}\delta H/\delta u\\ \delta H/\delta\varphi\end{pmatrix} \qquad B^2 = \left(\begin{array}{c|c} u\partial+\partial u+c\partial^3 & \varphi\partial+\partial\varphi/2 \\ \hline \partial\varphi+\varphi\partial/2 & 2s(u-\partial^2)\end{array}\right) \qquad 0\neq s\in\mathbb{C} \tag{16}$$

$$\begin{pmatrix}\dot{v}\\ \dot{\psi}\end{pmatrix} = B\begin{pmatrix}\delta H/\delta v\\ \delta H/\delta\psi\end{pmatrix} \qquad B = \left(\begin{array}{c|c} -c^{-1}\partial/4 & 0 \\ \hline 0 & 2s\end{array}\right) \tag{17}$$

$$u = -2c(v_x+v^2) + \psi\psi_x/4s \qquad \varphi = \psi_x + v\psi \tag{18}$$

where φ and ψ are new odd variables, B^2 and B are (super) Hamiltonian matrices, and (18) is a (super) Hamiltonian map (this can be checked by using Kupershmidt (1985c Theorem A 2.40). Passing to the Taylor representation of (16)-(18) (after we have checked that formula (7) remains true also for odd variables, which it does), we obtain

$$\dot{L}_n = \sum_{m\in\mathbb{Z}} (n-m)\partial\tilde{H}/\partial L_m + cn(n^2-1)\partial\tilde{H}/\partial L_{-n} + \sum_{q\in\frac{1}{2}+\mathbb{Z}} (-q+n/2)\Gamma_{n+q}\partial\tilde{H}/\partial\Gamma_q \tag{19a}$$

$$\dot{\Gamma}_p = \sum (p-m/2)\Gamma_{p+m}\partial\tilde{H}/\partial L_m + 2s\sum L_{p+q}\partial\tilde{H}/\partial\Gamma_q - 2s(p^2-\tfrac{1}{4})\partial\tilde{H}/\partial\Gamma_{-p} \tag{19b}$$

$$\dot{a}_n = -(4c)^{-1}n\partial\tilde{H}/\partial a_{-n} \qquad \dot{w}_p = 2s\partial\tilde{H}/\partial\omega_{-p} \tag{20}$$

$$L_n = -2c\left((n-1)a_n + \sum a_m a_{n-m}\right) + (4s)^{-1}\sum (q+\tfrac{1}{2})\omega_q v_{n+1-q} \tag{21a}$$

$$\Gamma_p = (p-\tfrac{1}{2})\omega_p + \sum a_m w_{p-m} \tag{21b}$$

where

$$\omega_p = \psi_{p-1/2} \qquad \Gamma_p = \varphi_{p-3/2}. \tag{22}$$

In the language of Poisson brackets, (19) and (20) can be rewritten in the form (14) and (15) plus, respectively,

$$\{L_n, \Gamma_q\} = (-q+n/2)\Gamma_{n+q} \qquad \{\Gamma_p, \Gamma_q\} = 2sL_{p+q} - 2s(p^2-\tfrac{1}{4})\delta_{p,-q} \tag{23}$$

$$\{a_n, \omega_p\} = 0 \qquad \{\omega_p, \omega_q\} = 2s\delta_{p,-q}. \tag{24}$$

Formulae (14) and (23) define the NSR superalgebra, as expected. Formulae (24) show that the bosonic oscillators do not interact with the fermionic ones. Formulae (21) provide an oscillator representation of the NSR superalgebra. It is curious to note that physicists originally found the quadratic term in (21*b*) by pure guesswork (see Schwarz 1973 formula (4.6)).

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